Caltech Division of the Humanities and Social Sciences

Integration and Differentiation

KC Border

Fall 1996

1 The Classical Fundamental Theorems

We start with a review of the Fundamental Theorems of Calculus, as presented in Apostol [2]. The notion of integration employed is the Riemann integral. Recall that a bounded function is Riemann integrable on an interval $[a, b]$ if and only it is continuous except on a set of Lebesgue measure zero. In this case its Lebesgue integral and its Riemann integral are the same.

Definition 1 *An* ***indefinite integral*** *F of f is a function such that for some a in I, the function F satisfies*

$$F(x) = \int_a^x f(s)\,ds \text{ for all } x \text{ in } I.$$

Different values of a give rise to different indefinite integrals of f.

An antiderivative is distinct from the concept of an indefinite integral.

Definition 2 *A function P is a* primitive *or* antiderivative *of a function f on an open interval I if*

$$P'(x) = f(x) \text{ for every } x \text{ in } I.$$

Leibniz' notation for this is $\int f(x)\,dx = P(x) + C$. Note that if P is an antiderivative of f, then so is $P + C$ for any constant function C.

Despite the similarity in notation, the statement that P is an antiderivative of f is a statement about the derivative of P, namely that $P'(x) = f(x)$ for all x in I; whereas the statement that F is an indefinite integral of f is a statement about the integral of f, namely that there exists some a in I with $\int_a^x f(s)\,ds = F(x)$ for all x in I. Nonetheless there is a close connection between the concepts, which justifies the similar notation. The connection is laid out in the two Fundamental Theorems of Calculus.

First Fundamental Theorem of Calculus [2, Theorem 5.1, p. 202] *Let f be integrable on $[a, x]$ for each x in $I = [a, b]$. Let $a \leqslant c \leqslant b$, and define the indefinite integral F of f by*

$$F(x) = \int_c^x f(s)\,ds.$$

Then F is differentiable at every x in (a, b) where f is continuous, and at such points $F'(x) = f(x)$.

That is, an indefinite integral of a continuous integrable function is also an antiderivative of the function.

This result is often loosely stated as, "the integrand is the derivative of its (indefinite) integral," which is not strictly true unless the integrand is continuous.

Second Fundamental Theorem of Calculus [2, Theorem 5.3, p. 205] *Let f be continuous on (a, b) and suppose that f possesses an antiderivative P. That is, $P'(x) = f(x)$ for every x in (a, b). Then for each x and c in (a, b), we have*

$$P(x) = P(c) + \int_c^x f(s)\,ds = P(c) + \int_c^x P'(s)\,ds.$$

That is, an antiderivative of a continuous function is also an indefinite integral.

This result is often loosely stated as, "a function is the (indefinite) integral of its derivative," which is not true. What is true is that "a function that happens to be an indefinite integral of something, is an (indefinite) integral of its derivative." To see this, suppose that F is an indefinite integral of f. That is, for some a the Riemann integral $\int_a^x f(s)\,ds$ is equal to $F(x)$ for every x in the interval I. In particular, f is Riemann integrable over $[a, x]$, so it is continuous everywhere in I except possibly for a set N of Lebesgue measure zero. Consequently, by the First Fundamental Theorem, except possibly for a set N of measure zero, F' exists and $F'(x) = f(x)$. Thus the Lebesgue integral of F' over $[a, x]$ exists for every x and is equal to the Riemann integral of f over $[a, x]$, which is equal to $F(x)$. In that sense, F is the integral of its derivative. Thus we see that a necessary condition for a function to be an indefinite integral is that it be differentiable almost everywhere.

Is this condition sufficient as well? It turns out that the answer is no. There exist continuous functions that are differentiable almost everywhere that are not an indefinite integral of their derivative. Indeed such a function is not an indefinite integral of any function. The commonly given example is the Cantor ternary function.

2 The Cantor ternary function

Given any number x with $0 \leqslant x \leqslant 1$ there is an infinite sequence $a_1, a_2, \ldots$, where each a_n belongs to $\{0, 1, 2\}$ such that $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$. This sequence is called the **ternary**

representation of x. If x is of the form $\frac{N}{3^m}$ (in lowest terms), then it has two ternary representations: $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$, where $a_m > 0$ and $a_n = 0$ for $n > m$, and another representation of the form $x = \sum_{n=1}^{m-1} \frac{a_n}{3^n} + \frac{a_m - 1}{3^m} + \sum_{n=m+1}^{\infty} \frac{2}{3^n}$. But these are the only cases of a nonunique ternary representation, and there are only countably many such numbers. (See, e.g., Boyd [3, Theorem 1.23, p. 20].)

Given $x \in [0,1]$, let $N(x)$ be the first n such that $a_n = 1$ in the ternary representation. If x has two ternary representations use the one that gives the larger value of $N(x)$. If x has a ternary representation with no $a_n = 1$, then $N(x) = \infty$. The **Cantor set** $\mathcal{C}$ consists of all numbers x in $[0,1]$ for which $N(x) = \infty$. That is, those that have a ternary representation where no $a_n = 1$. That is, all numbers x of the form $x = \sum_{n=1}^{\infty} \frac{2b_n}{3^n}$, where each b_n belongs to $\{0,1\}$. Each distinct sequence of 0s and 1s gives rise to a distinct element of $\mathcal{C}$. Indeed some authors identify the Cantor set with $\{0,1\}^{\mathbb{N}}$ endowed with its product topology, since the mapping $(b_1, b_2, \ldots) \mapsto \sum_{n=1}^{\infty} \frac{2b_n}{3^n}$ is a homeomorphism. Also note that a sequence $(b_1, b_2, \ldots)$ of 0s and 1s also corresponds to a unique subset of $\mathbb{N}$, namely $\{n \in \mathbb{N} : b_n = 1\}$. Thus there are as many elements $\mathcal{C}$ as there are subset of $\mathbb{N}$, so the Cantor set is uncountable. (This follows from the Cantor diagonal procedure.) Yet the Cantor set includes no interval.

It is perhaps easier to visualize the complement of the Cantor set. Let

$$\mathcal{A}_n = \{x \in [0,1] : N(x) = n\}.$$

The complement of the Cantor set is $\bigcup_{n=1}^{\infty} \mathcal{A}_n$. Define

$$\mathcal{C}_n = [0,1] \setminus \bigcup_{k=1}^{n} \mathcal{A}_k,$$

so that $\mathcal{C} = \bigcap_{n=0}^{\infty} \mathcal{C}_n$. Now $\mathcal{A}_1$ consists of those x for which $a_1 = 1$ in its ternary expansion. This means that

$$\mathcal{A}_1 = \left(\tfrac{1}{3}, \tfrac{2}{3}\right) \qquad \text{and} \qquad \mathcal{C}_1 = \left[0, \tfrac{1}{3}\right] \cup \left[\tfrac{2}{3}, 1\right].$$

Note that $N(\frac{1}{3}) = \infty$ since $\frac{1}{3}$ can also be written as $\sum_{n=2}^{\infty} \frac{2}{3^n}$, so $a_1 = 0$, $a_n = 2$ for $n > 1$. Now $\mathcal{A}_2$ consists of those x for which $a_1 = 0$ or $a_1 = 2$ and $a_2 = 1$ in its ternary expansion. Thus

$$\mathcal{A}_2 = \left(\tfrac{1}{9}, \tfrac{2}{9}\right) \cup \left(\tfrac{7}{9}, \tfrac{8}{9}\right) \qquad \text{and} \qquad C_2 = \left[0, \tfrac{1}{9}\right] \cup \left[\tfrac{2}{9}, \tfrac{1}{3}\right] \cup \left[\tfrac{2}{3}, \tfrac{7}{9}\right] \cup \left[\tfrac{8}{9}, 1\right].$$

Each $\mathcal{C}_n$ is the union of 2^n closed intervals, each of length $\frac{1}{3^{n-1}}$, and $\mathcal{A}_{n+1}$ consists of the open middle third of each of the intervals in $\mathcal{C}_n$. The total length of the removed open segments is

$$\frac{1}{3} + 2 \cdot \frac{1}{9} + 4 \cdot \frac{1}{27} + \cdots = \sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{3} \cdot \frac{1}{1 - \frac{2}{3}} = 1.$$

Thus the total length of the Cantor set is $1 - 1 = 0$.

The Cantor ternary function f is defined as follows. On the open middle third $(\frac{1}{3}, \frac{2}{3})$ its value is $\frac{1}{2}$. On the open interval $(\frac{1}{9}, \frac{2}{9})$ its value is $\frac{1}{4}$ and on $(\frac{7}{9}, \frac{8}{9})$ its value is $\frac{3}{4}$. Continuing in this fashion, the function is defined on the complement of the Cantor set. The definition is extended to the entire interval by continuity. See Figure 1. A more

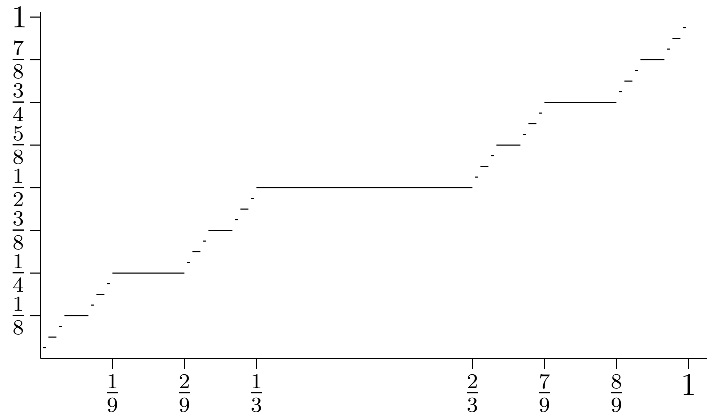

Figure 1. Partial graph of the Cantor ternary function.

precise but more opaque definition is this:

$$f(x) = \begin{cases} \sum_{n=1}^{N(x)-1} \frac{\frac{1}{2}a_n}{2^n} + \frac{a_{N(x)}}{2^{N(x)}} & \text{if } N(x) < \infty, \\ \sum_{n=1}^{\infty} \frac{\frac{1}{2}a_n}{2^n} & \text{if } N(x) = \infty. \end{cases}$$

In any event notice that f is constant on each open interval in some $\mathcal{A}_n$, so it is differentiable there and $f' = 0$. Thus f is differentiable almost everywhere, and $f' = 0$ wherever it exists, but

$$f(1) - f(0) = 1 \quad \text{and} \quad \int_0^1 f'(x)\, dx = 0.$$

3 Integration by Parts

The fundamental theorems of calculus enable us to prove the following result.

Integration by Parts *Suppose f and g are continuously differentiable on the open interval I. Let $a < b$ belong to I. Then*

$$\int_a^b f(x)g'(x)\, dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\, dx.$$

Proof based on Apostol [2, Section 5.9, pp. 217–218]: Define $h(x) = f(x)g(x)$. Then h is continuously differentiable on I and $h'(x) = f(x)g'(x) + f'(x)g(x)$. That is, h is an antiderivative of the continuous function $f(x)g'(x) + f'(x)g(x)$. So by the Second Fundamental Theorem of Calculus

$$h(b) - h(a) = \int_a^b f(x)g'(x) + f'(x)g(x)\,dx,$$

and rearranging terms gives the conclusion. ∎

4 A More general result

To apply the Second Fundamental Theorem of Calculus, we need $f'g + g'f$ to be a continuous function. The only reasonable sufficient condition for this is that f and g be continuously differentiable. However, Fubini's Theorem allows us to prove the integration by parts formula under weaker conditions. All we need is that f and g be indefinite integrals. That is, we do not need f and g to be differentiable everywhere, only that they are indefinite integrals. More formally we have:

Integration by Parts, Part II *Suppose F and G satisfy*

$$F(x) = F(a) + \int_a^x f(s)\,ds$$

and

$$G(x) = G(a) + \int_a^x g(s)\,ds$$

for every x in $[a,b]$, where f and g are integrable over $[a,b]$ and fg is integrable over $[a,b] \times [a,b]$. Then

$$\int_a^b F(x)g(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\,dx.$$

Proof based on Fubini's Theorem: Write

$$\begin{aligned}
\int_a^b F(x)g(x)\,dx &= \int_a^b \left(F(a) + \int_a^x f(s)\,ds\right) g(x)\,dx \\
&= \int_a^b F(a)g(x)\,dx + \int_a^b \left(\int_a^x f(s)g(x)\,ds\right) dx \\
&= F(a)\big(G(b) - G(a)\big) + \int_a^b \int_a^b \mathbf{1}_{[s \leqslant x]} f(s)g(x)\,d(s,x) \\
&= F(a)\big(G(b) - G(a)\big) + \int_a^b f(s) \left(\int_a^b \mathbf{1}_{[x \geqslant s]} g(x)\,dx\right) ds \\
&= F(a)\big(G(b) - G(a)\big) + \int_a^b f(s) \left(\int_s^b g(x)\,dx\right) ds \\
&= F(a)\big(G(b) - G(a)\big) + \int_a^b f(s)\big(G(b) - G(s)\big)\,ds \\
&= F(a)\big(G(b) - G(a)\big) + \int_a^b f(s)G(b)\,ds - \int_a^b f(s)G(s)\,ds \\
&= F(a)\big(G(b) - G(a)\big) + G(b)\big(F(b) - F(a)\big) - \int_a^b f(s)G(s)\,ds \\
&= F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\,dx,
\end{aligned}$$

where

$$\mathbf{1}_{[s \leqslant x]} = \mathbf{1}_{[x \geqslant s]} = \begin{cases} 1 & s \leqslant x \\ 0 & s > x. \end{cases}$$

∎

5 A Still More General Result

5.1 Finite measures and nondecreasing functions

Let μ be a finite (nonnegative) measure on the Borel subsets of $\boldsymbol{R}$. Define the function $F_\mu \colon \boldsymbol{R} \to \boldsymbol{R}_+$ by

$$F_\mu(x) = \mu\big(\{y \in \boldsymbol{R} : y \leqslant x\}\big).$$

F_μ is called the distribution function of μ, and has the following properties:

1. F_μ is nondecreasing.
2. F_μ is right continuous. That is, $F_\mu(x) = \lim_{y \downarrow x} F_\mu(y)$.
3. $\lim_{x \to -\infty} F_\mu(x) = 0$.

4. $\lim_{x\to\infty} F_\mu(x) = \mu(\boldsymbol{R})$.
5. $F(b) - F(a) = \mu\big((a,b]\big)$ for $a \leqslant b$.

Conversely, for any $F\colon \boldsymbol{R} \to \boldsymbol{R}_+$ satisfying

1. F is nondecreasing.
2. F is right continuous.
3. $\lim_{x\to-\infty} F_(x) = 0$.
4. $\lim_{x\to\infty} F_(x) = a < \infty$.

there is a unique nonnegative Borel measure μ_f satisfying $\mu_F\big((a,b]\big) = F(b) - F(a)$ for $a \leqslant b$. Given a distribution function $F\colon \boldsymbol{R} \to \boldsymbol{R}_+$ and a μ_F-integrable function f, the *Lebesgue–Stieltjes integral*

$$\int f\,dF = \int f\,d\mu_f$$

by definition.

Integration by Parts for Distribution Functions *Let F and G be distribution functions on $\boldsymbol{R}$. Then*

$$\int_{(a,b]} F(x)\,dG(x) = F(b)G(b) - F(a)G(a) - \int_{(a,b]} G(x^-)\,dF(x), \tag{1}$$

where $G(x^-) = \lim_{y\uparrow x} G(y)$.

Proof: Define $A = \{(x,y) \in (a,b]^2 : x \leqslant y\}$. By Fubini's Theorem 6 on iterated integrals, we have

$$\begin{aligned}
&\textstyle\int\int \mathbf{1}_A\,d(\mu_G \times \mu_F) \\
&\qquad = \textstyle\int_{(a,b]} \left(\int_{(a,b]} \mathbf{1}_A\,d\mu_F\right) d\mu_G \quad = \quad \int_{(a,b]} (F(x) - F(a))\,d\mu_G(x) \\
&\qquad = \textstyle\int_{(a,b]} \left(\int_{(a,b]} \mathbf{1}_A\,d\mu_G\right) d\mu_F \quad = \quad \int_{(a,b]} (G(b) - G(y^-))\,d\mu_F(y),
\end{aligned}$$

where $\mathbf{1}_A$ is the indicator function defined by $\mathbf{1}_A(x,y) = \begin{cases} 1 & (x,y) \in A \\ 0 & (x,y) \notin A \end{cases}$.

Rearrange to get

$$\int_{(a,b]} (F(x) - F(a))\,d\mu_G(x) = \int_{(a,b]} \big(G(b) - G(y^-)\big)\,d\mu_F(y)$$

or

$$\int_{(a,b]} F(x)\,dG(x) - F(a)\big(G(b) - G(a)\big) = G(b)\big(F(b) - F(a)\big) - \int_{(a,b]} G(y^-)\,d\mu_F(y),$$

from which the conclusion follows. ∎

Corollary 1 *If either F or G is continuous, then*

$$\int_{[a,b]} F(x)\,dG(x) = F(b)G(b) - F(a)G(a) - \int_{[a,b]} G(x)\,dF(x).$$

Corollary 2 *Let F be a cumulative distribution function with $F(0) = 0$, that is, the cumulative distribution function of a nonnegative random variable Then for any $p > 0$,*

$$\int_{[0,\infty)} x^p\,dF(x) = p\int_0^\infty \big(1 - F(x)\big)x^{p-1}\,dx$$

Proof: Fix $b > 0$ and set

$$G_b(x) = \begin{cases} 0 & x \leqslant 0 \\ x^b & 0 \leqslant x \leqslant b \\ b^p & x \geqslant b \end{cases}$$

and note that G_b is a continuous distribution function. By Corollary 1,

$$\begin{aligned} \int_0^b x^p\,dF &= F(b)b^p - \int_0^b F(x)\,dG_b(x). \\ &= F(b)b^p - p\int_0^b F(x)x^{p-1}\,dx \\ &= p\int_0^b (F(b) - F(x))x^{p-1}\,dx, \end{aligned}$$

since G_b has derivative px^{p-1} on $(0, b)$. Now let $b \to \infty$. ∎

6 Fubini's Theorem

There is a collection of related results that are all referred to as Fubini's theorem. This version is taken from Halmos [4, Theorem C, p. 148]. See also Aliprantis and Burkinshaw [1, Theorem 26.6, p. 212] or Royden [5, Theorem 12.19, p. 307].

Fubini's Theorem *Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ be σ-finite measure spaces. If the function $f\colon X \times Y \to \boldsymbol{R}$ is $\mu \times \nu$-integrable, then the fuctions $x \mapsto \int_Y f(x, y)\,d\nu(y)$ and $y \mapsto \int_X f(x, y)\,d\mu(x)$ are μ-integrable and ν-integrable respectively, and*

$$\int_{X\times Y} f(x,y)\,d(\mu\times\nu)(x,y) = \int_X \left(\int_Y f(x,y)\,d\nu(y)\right) d\mu(x) = \int_Y \left(\int_X f(x,y)\,d\mu(x)\right) d\nu(y).$$

References

[1] Aliprantis, C. D. and O. Burkinshaw. 1998. *Principles of real analysis*, 3d. ed. San Diego: Academic Press.

[2] Apostol, T. M. 1967. *Calculus*, 2d. ed., volume 1. Waltham, Massachusetts: Blaisdell.

[3] Boyd, D. Classical analysis, volume 1. Notes prepared for Ma 108 abc. Published occasionally since at least 1972 by the Department of Mathematics, California Institute of Technology, 253-37, Pasadena CA 91125.

[4] Halmos, P. R. 1974. *Measure theory*. Number 18 in Graduate Texts in Mathematics. New York: Springer–Verlag. Reprint of the edition published by Van Nostrand, 1950.

[5] Royden, H. L. 1988. *Real analysis*, 3d. ed. New York: Macmillan.