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Brief notes on Revenue Equivalence

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The Revenue Equivalence Theorem states roughly that in the risk-neutral IID private values case, the seller's ex ante expected revenue depends only on the reduced form allocation rule. We will prove the theorem by deriving a formula for the ex ante expected revenue.

1 The risk-neutral IID private values environment

This is the case that Riley and Samuelson [8] call the IID case and McAfee and McMillan [6] call the benchmark case.

- There is a single seller, with one unit to sell.
- The seller has a cash value v_0 for the object.
- There are n (potential) buyers or bidders. Bidders attach a cash value v to possessing the object. Each bidder's value is drawn independently from the interval $[\underline{v}, \overline{v}]$ according to a probability with cumulative distribution F that satisfies $F(\underline{v}) = 0$, $F(\overline{v}) = 1$, and it is assumed that $0 \leqslant \underline{v}$, F is continuous, and that F has a strictly positive first derivative (density). This implies the probability that two buyers have the same value is zero.
- The buyers are risk neutral and the utility of a bidder with value v is $v - p$ if he has the object and pays p, and $0 - p$ if he pays p without winning the object. Thus the a bidder's ex ante expected utility is

  $$v \times (\text{probability of ``winning''}) - (\text{ex ante expected payment}).$$

  Thus the utility of not participating in the auction is zero.

2 Definition of an auction and the Revelation Principle

We invoke the Revelation Principle to say that

- a **symmetric auction** is characterized by a pair of functions

$$P\colon [\underline{v}, \overline{v}] \to \boldsymbol{R}, \quad \text{and} \quad Q\colon [\underline{v}, \overline{v}] \to [0, 1],$$

  where $P(v)$ is the ex ante expected payment of a bidder with value v and $Q(v)$ is the reduced form allocation rule, that is, the probability that a bidder with value v wins the object.[1]

- Not every function $Q\colon [\underline{v}, \overline{v}] \to [0, 1]$ is a feasible reduced form allocation rule. See Appendix B for restrictions. It also turns out that the payment function P is redundant, but we'll see that in a moment.

- Moreover, we may assume that the auction (P, Q) is **incentive compatible**. That is, defining the payoff to reporting x when your value is v,

$$\Pi(v, x) = vQ(x) - P(x), \tag{1}$$

  we have, for all $x, v \in [\underline{v}, \overline{v}]$, the **incentive compatibility** condition,

$$\Pi(v, v) \geqslant \Pi(v, x). \tag{\textbf{IC}}$$

This says that a bidder maximizes his ex ante expected utility by announcing his true value. This has the following consequence. Define the indirect utility

$$U(v) = \Pi(v, v) = vQ(v) - P(v), \tag{2}$$

and for each x define the affine function $\ell_x\colon [\underline{v}, \overline{v}] \to \boldsymbol{R}$ by

$$\ell_x(v) = Q(x)v - P(x).$$

Then (**IC**) implies

$$U(v) = \sup_{x \in [\underline{v}, \overline{v}]} \ell_x(v) = \ell_v(v).$$

It turns out this has a number of incredibly useful consequences (see my notes on convex analysis). In particular,

- ◦ U is convex and continuous on $[\underline{v}, \overline{v}]$.

[1] Harris and Raviv [3] show that it is enough to consider symmetric reduced form auctions in the IID case.

- $Q(v)$ is a subgradient of U at v for every v, so
- Q is nondecreasing,
- $U'(v) = Q(v)$ whenever U' exists, which is everywhere except possibly on a countable set, and
- for any $a, v \in [\underline{v}, \overline{v}]$,

$$U(v) = U(a) + \int_a^v Q(x)\,dx. \tag{3}$$

- Define the **cutoff** v_* of the auction by

$$v_* = \sup\{x : Q(x) = 0\}.$$

  It follows from (3) and the fact that Q is nonnegative and nondecreasing, that U is constant on $[\underline{v}, v_*]$, and strictly increasing on $[v_*, \overline{v}]$.

3 Voluntary participation

We cannot force a bidder to participate in the auction, so we must have $U(v) \geqslant 0$ for all v. In particular, $U(\underline{v}) = U(v_*) \geqslant 0$. If the seller maximizes revenue, then this constraint will bind and we will have $U(\underline{v}) = U(v_*) = 0$. More generally, each buyer may have an **outside option** that gives him utility $\underline{u} > 0$, but we shall stay with the case $\underline{u} = 0$.

4 Efficiency

Following Riley and Samuleson [8], we now impose another restriction on our auction, namely,

- if a bidder wins the object it must be the bidder with the highest value. This implies that if $Q(v) > 0$, then

$$Q(v) = F^{n-1}(v). \tag{\textbf{E}}$$

  (Remember, continuity of F gives tie values probability zero.)

We show in Appendix B that this is a feasible reduced form allocation rule.

5 Seller's ex ante expected revenue

1 Revenue Equivalence Theorem (Riley–Samuelson) *Under the conditions discussed above, the seller's ex ante expected revenue from an auction with cutoff v_* is*

$$n \int_{v_*}^{\overline{v}} \Big(vF'(v) + F(v) - 1 \Big) F^{n-1}(v) \, dv. \tag{\textbf{R}}$$

An important thing to note about (**R**) is that while it depends on the function Q ($= F^{n-1}$), and the cutoff $v_* = \sup\{x : Q(x) = 0\}$, the payment function P appears nowhere.

Proof: By (2) and (3), for $v > v_*$ we have

$$P(v) = vQ(v) - U(v) = vF^{n-1}(v) - \int_{v_*}^{v} F^{n-1}(x) \, dx.$$

(Recall that $Q = 0$ is constant on $[\underline{v}, v_*]$.)

Since the seller's revenue is the sum of the buyers' payments, the ex ante expected revenue must be n times the ex ante expected value of P. That is,

$$\begin{aligned} \text{ex ante expected revenue} &= n \int_{\underline{v}}^{\overline{v}} P(v) F'(v) \, dv \\ &= n \int_{v_*}^{\overline{v}} vQ(v) F'(v) \, dv - n \int_{v_*}^{\overline{v}} U(v) F'(v) \, dv \\ &= n \int_{v_*}^{\overline{v}} vF^{n-1}(v) F'(v) \, dv - n \int_{v_*}^{\overline{v}} U(v) F'(v) \, dv. \end{aligned}$$

We can get a hamdle on the last integral by observing that

$$\int_{v_*}^{\overline{v}} U(v) F'(v) \, dv = \int_{v_*}^{\overline{v}} \left(\int_{v_*}^{v} F^{n-1}(x) \, dx \right) F'(v) \, dv = \int_{v_*}^{\overline{v}} F^{n-1}(v) \Big(1 - F(v) \Big) \, dv, \tag{4}$$

where the final equality is an application of Fubini's Theorem, Lemma 3 below. Substituting this above yields (**R**).

■

There is another interpretation of (**R**). We know the object will go to the participating buyer with the highest value. The ex ante expected value of this bidder's value is gotten by integrating v with respect to the density of the maximum order statistic, namely $nF^{n-1}(v)F'(v)$, over the interval of participating values $[v_*, \overline{v}]$. So the ex ante expected highest value is

$$n \int_{v_*}^{\overline{v}} vF^{n-1}(v) F'(v) \, dv, \tag{5}$$

which is the first term of ($\mathbf{R}$). But the seller cannot keep this. Ex ante he expects to pay each buyer

$$\int_{\underline{v}}^{\overline{v}} U(v)F'(v)\,dv.$$

Taking (5) and subtracting n times this payment leaves the seller with ($\mathbf{R}$).

Here is yet one more interpretation. We know that the second-price auction has the property that the highest-value bidder wins but pays the second highest price. This is true even with a cutoff. Since the highest-value bidder wins, the auction is efficient so the seller's ex ante expected revenue is given by ($\mathbf{R}$). That means that ($\mathbf{R}$) must also be the expected value of the second-highest value. Lemma 2 in Appendix A below confirms this for the case $v_* = \underline{v}$.

The outside option

If we have scaled buyers' utility so that each has an outside opportunity $\underline{u}$, then the seller must reduce P by $\underline{u}$, reducing his revenue by $n\underline{u}$.

6 Maximizing the seller's utility

Since the ex ante expected revenue from an auction with cutoff v_* is given by ($\mathbf{R}$) the seller can maximize his ex ante expected utility by choosing v_* to maximize

$$v_0 F^n(v_*) + n\int_{v_*}^{\overline{v}} \Big(vF'(v) + F(v) - 1\Big)F^{n-1}(v)\,dv,$$

where, as you recall v_0 is the seller's value for the object and $F^n(v_*)$ is the probability that no buyer meets the cutoff, so the seller keeps the object.

The first order condition for this maximization is that at the optimal cutoff v_*^*, we have

$$v_0 n F^{n-1}(v_*^*)F'(v_*^*) - n\Big(v_*^* F'(v_*^*) + F(v_*^*) - 1\Big)F^{n-1}(v_*^*) = 0$$

provided $\underline{v} < v_*^* < \overline{v}$. This implies that v_*^* solves

$$v_*^* - \frac{1 - F(v_*^*)}{F'(v_*^*)} = v_0.$$

Note that this seller-optimal cutoff does not depend on the number of bidders!

Note that this applies if the seller-optimal v_*^* is interior. If $v_*^* = \underline{v}$, then all we can conclude is that $v_0 \leqslant \underline{v} - \frac{1-F(\underline{v})}{F'(\underline{v})}$. Likewise, it may be optimal to exclude all buyers and just keep the object, in which case $v_0 \geqslant \overline{v} - \frac{1-F(\overline{v})}{F'(\overline{v})}$.

7 Virtual values

The function $\tau(v)$ defined by

$$\tau(v) = v - \frac{1 - F(v)}{F'(v)}$$

is called the **virtual value** function. We have just seen that the seller-optimal interior cutoff satisfies

$$\tau(v_*^*) = v_0.$$

If the virtual value function is invertible, then v_*^* is the unique solution to his equation, that is,

$$v_*^* = \tau^{-1}(v_0).$$

Virtual values also shed more light on the ex ante expected revenue (**R**). Write

$$\begin{aligned} n\int_{v_*}^{\overline{v}} \Big(vF'(v) + F(v) - 1\Big) F^{n-1}(v)\,dv &= n\int_{v_*}^{\overline{v}} \left(v - \frac{1-F(v)}{F'(v)}\right) F^{n-1}(v)F'(v)\,dv \\ &= \int_{v_*}^{\overline{v}} \tau(v)\frac{d}{dv}F^n(v)\,dv \\ &= \boldsymbol{E}\,\tau\Big(\boldsymbol{v}_{(1)}\Big)\mathbf{1}_{[v_*,\overline{v}]}\Big(\boldsymbol{v}_{(1)}\Big). \end{aligned} \tag{6}$$

Recall that $\frac{d}{dv}F^n(v)$ is the density of the maximum order statistic, so the ex ante expected revenue is the ex ante expected value of the virtual type of the highest type bidder. In terms of ex ante expectation, the virtual value of a bidder is that bidder's value's contribution to the seller's revenue.

If the highest-value buyer has value v, the seller cannot extract all that as ex ante revenue, instead he has to let the buyer keep $\frac{1-F(v)}{F'(v)}$. For that reason this quantity is sometimes called the **informational rent** accruing to a bidder with value v, but a better term might be **buyer's surplus**.[2]

8 Dropping efficiency

We imposed the requirement that the highest-value bidder wins the auction, which implies that $Q(v) = F^{n-1}(v)$. What if we drop that requirement and merely require that Q be an incentive compatible feasible reduced form allocation rule? It still follows as above, assuming $U(\underline{v}) = \underline{u} = 0$, that

$$P(v) = vQ(v) - \int_{\underline{v}}^{v} Q(x)\,dx.$$

[2]Much of auction theory was developed at business schools, so the theorists identified with the monopoly seller as the "good guy." Thus the term "optimal auction" usually means optimal for the monopoly seller. Likewise, the term "informational rent" is used to cast the buyers in the role of "bad guy" monopsonists in their private information. We already had more neutral terms of producer's and consumers' surplus.

(We can use $\underline{v}$ as the lower limit of integration, as $Q(x) = 0$ for $v < v_* = \sup\{x : Q(x) = 0\}$. Thus the ex ante expected revenue depends only on the function Q—two auctions with the same reduced form Q are revenue equivalent, and the ex ante expected revenue is

$$n \int_{\underline{v}}^{\overline{v}} P(v)F'(v)\,dv = n \int_{\underline{v}}^{\overline{v}} \left(vQ(v) - \int_{\underline{v}}^{v} Q(x)\,dx \right) F'(v)\,dv$$
$$= n \int_{\underline{v}}^{\overline{v}} vQ(v)F'(v) - Q(v)\big(1 - F(v)\big)\,dv,$$

again by Lemma 3. But now we can rewrite the last expression as

$$n \int_{\underline{v}}^{\overline{v}} \tau(v)Q(v)F'(v)\,dv, \tag{7}$$

which reduces to (6) in the efficient case. The seller's problem is thus to choose the function Q to maximize (7) subject to the constraint that Q be nondecreasing and a feasible reduced forma allocation rule. It is actually easier to work not with the reduced form allocation rule, but with the direct allocation rule. See, for instance, Myerson [7].

9 First-price auctions

So what does this have to do with first-price auctions? Well if the Nash equilibrium of a first-price auction has the property that the bidding strategies are nondecreasing functions, then the bidder with the highest value "wins" the auction, and it is thus Riley–Samuelson-efficient, and the revenue is given by Theorem 1. Likewise, the second-price sealed-bid auction has a dominant strategy equilibrium of truthful revelation, so the highest-value bidder "wins," and the revenue is again given by Theorem 1. Thus any nondecreasing equilibrium bidding strategy is revenue equivalent to the second-price auction.

So the question becomes, when does the first-price auction have a nondecreasing equilibrium bidding strategy? One answer, given by Myerson [7] is when the cdf F is uniform and the virtual value function τ is increasing.

Appendices

A Order statistics

Let $\boldsymbol{v}$ be a random vector in $[\underline{v}, \overline{v}]^n$ where the components are IID according to the cdf F. For the purposes of these notes, the n^{th} order statistic, or maximum of the components of $\boldsymbol{v}$, denoted $\boldsymbol{v}_{(1)}$,[3] has cdf F^n, and hence density $nF^{n-1}F'$. The $n-1^{\text{st}}$ order statistic, or second greatest value of the components of $\boldsymbol{v}$, denoted $\boldsymbol{v}_{(2)}$, has cdf $nF^{n-1}-(n-1)F^n$, and density $n(n-1)F^{n-2}(1-F)F'$.

The expected value of $\boldsymbol{v}_{(2)}$ is given in the next lemma.

2 Lemma *Assuming F is differentiable with $F(\underline{v}) = 0$, and $F(\overline{v}) = 1$,*

$$\begin{aligned} \boldsymbol{E}\,\boldsymbol{v}_{(2)} &= n\int_{\underline{v}}^{\overline{v}} \Big(vF'(v) + F(v) - 1\Big)F^{n-1}(v)\,dv \\ &= n\int_{\underline{v}}^{\overline{v}} \left(v - \frac{1-F(v)}{F'(v)}\right) F^{n-1}(v)F'(v)\,dv. \end{aligned}$$

Proof: Integrate v with respect to the density of $\boldsymbol{v}_{(2)}$ and integrate by parts:

$$\begin{aligned} \boldsymbol{E}\,\boldsymbol{v}_{(2)} &= n(n-1)\int_{\underline{v}}^{\overline{v}} v\Big(1-F(v)\Big)F^{n-2}(v)F'(v)\,dv \\ &= n\int_{\underline{v}}^{\overline{v}} \underbrace{v\Big(1-F(v)\Big)}\,\underbrace{\frac{d}{dv}F^{n-1}(v)}\,dv \\ &= n\left\{\underbrace{v\Big(1-F(v)\Big)F^{n-1}(v)\Big|_{\underline{v}}^{\overline{v}}}_{=0-0} - \int_{\underline{v}}^{\overline{v}} \left(1-F(v)-vF'(v)\right)F^{n-1}(v)\,dv\right\}, \end{aligned}$$

as desired. ∎

B Reduced form allocation rules

I mentioned earlier that not every function $Q\colon [\underline{v},\overline{v}] \to [0,1]$ can be a reduced form allocation rule. For instance, the function $Q(v) = 1$ for all v cannot be a reduced form because it says that every bidder will get the object with probability 1, which is impossible if $n > 1$. To be more careful we should start by realizing that what we get from the Revelation Principle is a direct revelation **allocation rule**

$$\boldsymbol{q}\colon [\underline{v},\overline{v}]^n \to [0,1]^n,$$

which determines the probability $\boldsymbol{q}_i(\boldsymbol{v})$ that bidder i is awarded (“wins”) the object when the vector of values is $\boldsymbol{v}$. The function $\boldsymbol{q}$ must satisfy

$$\boldsymbol{q}_i(\boldsymbol{v}) \geqslant 0, \qquad \boldsymbol{v} \in [\underline{v},\overline{v}]^n,\ i = 1,\dots,n$$

[3]Traditionally $\boldsymbol{v}_{(1)}$ would denote the first order statistic, that is, the minimum, but for auction theory it is more convenient to start at the maximum rather than the minimum.

and the feasibility condition

$$\sum_{i=1}^{n} \boldsymbol{q}_i(\boldsymbol{v}) \leqslant 1, \qquad \boldsymbol{v} \in [\underline{v}, \overline{v}]^n.$$

The **reduced form** of $\boldsymbol{q}$ is given by

$$Q_i(v) = \int_{[\underline{v},\overline{v}]^{n-1}} \boldsymbol{q}_i(\boldsymbol{v}_{-i}, v)\, dF^{n-1}(\boldsymbol{v}_{-i})$$

Harris and Raviv [3] show that it suffices to consider the case where all the Q_i are identical.[4]

Matthews [5] shows that an nondecreasing Q is a reduced form of a symmetric direct revelation auction in the IID private values environment if and only if it satisfies the Maskin–Riley–Matthews Condition

$$n \int_{v}^{\overline{v}} Q(x) F'(x)\, dx \leqslant 1 - F^n(v), \quad \text{for all } v \in [\underline{v}, \overline{v}]. \tag{\textbf{MRM}}$$

Border [1, 2] generalizes this to allow for more general "types" of bidders.

A consequence of this is that $Q(v) = F^{n-1}(v)$ is a valid reduced form allocation rule, as

$$n \int_{v}^{\overline{v}} F^{n-1}(x) F'(x)\, dx = \int_{v}^{\overline{v}} \frac{d}{dx} F^n(x)\, dx = F^n(\overline{v}) - F^n(v) = 1 - F^n(v).$$

C A lemma on integration

The following lemma is a straightforward application of Fubini's Theorem on interchanging the order of integration. (See, e.g., Royden [9, Theorem 19, pp.307–308].)

3 Lemma *Let $g\colon [\underline{v}, \overline{v}] \to \boldsymbol{R}$ be integrable, and let $F\colon [\underline{v}, \overline{v}] \to [0,1]$ be a differentiable cdf. Then for any $v_* \in [\underline{v}, \overline{v}]$,*

$$\int_{v_*}^{\overline{v}} \left(\int_{v_*}^{v} g(x)\, dx \right) F'(v)\, dv = \int_{v_*}^{\bar{v}} g(v) \Big(1 - F(v)\Big)\, dv.$$

Proof: Let $\mathbf{1}_{x \leqslant v}$ denote the indicator function of the set of $\{(x,v) : x \leqslant v\}$. Then

$$\begin{aligned}
\int_{v_*}^{\overline{v}} \left(\int_{v_*}^{v} g(x)\, dx \right) F'(v)\, dv &= \int_{v_*}^{\overline{v}} \left(\int_{v_*}^{\overline{v}} \mathbf{1}_{x \leqslant v} g(x) F'(v)\, dx \right) dv \\
&= \int_{v_*}^{\overline{v}} \left(\int_{v_*}^{\overline{v}} \mathbf{1}_{x \leqslant v} F'(v)\, dv \right) g(x)\, dx \\
&= \int_{v_*}^{\overline{v}} (1 - F(x))\, g(x)\, dx \\
&= \int_{v_*}^{\bar{v}} g(x) \Big(1 - F(x)\Big)\, dx.
\end{aligned}$$

[4] The argument runs something like this. If the functions are not identical, symmetrize the auction by randomly assign the bidders numbers from 1 to n. Then the incentive constraints still hold and the seller is indifferent. This amounts to replacing each Q_i with an equally weighted convex combination.

Now replace the dummy variable x by v. ∎

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