These notes are far from completed.

1 Household problem

At time \( t \) the typical household receives real wage income \( w(t) \) and owns assets \( k(t) \), which produces income at the interest rate \( r(t) \) and depreciates at the rate \( n \) (due to population growth), and chooses consumption level \( c(t) \).

The continuous time budget can be written as

\[
\dot{k}(t) = w(t) + r(t)k(t) - c(t) - nk(t). \tag{1}
\]

The objective is to maximize

\[
\int_0^\infty u(c(t))e^{-(\rho-n)t} \, dt
\]

where \( \rho > n \). The present value Hamiltonian is

\[
H(k,c,t,p) = u(c)e^{-(\rho-n)t} + p\left(w(t) + r(t)k - c - nk\right)
\]

The first-order necessary conditions are

\[
\dot{k} = D_pH = w + r k - c - nk \tag{2}
\]

\[
\dot{p} = -D_kH = -(r - n)p \tag{3}
\]

\[
D_cH = u'(c(t))e^{-(\rho-n)t} - p(t) = 0 \tag{4}
\]

\[
\lim_{t \to \infty} p(t)k(t) = 0 \tag{5}
\]
Equation (4) implies
\[ p(t) = u'(c(t))e^{-(\rho-n)t} \]
\[ \dot{p}(t) = u''(c(t))\dot{c}(t)e^{-(\rho-n)t} - (\rho-n)u'(c(t))e^{-(\rho-n)t} \]
\[ \frac{\dot{p}}{p} = \frac{u''(c(t))\dot{c}(t)e^{-(\rho-n)t}}{u'(c(t))e^{-(\rho-n)t}} - (\rho-n) \]
\[ = \left( \frac{u''}{u'} \right) \frac{\dot{c}}{c} - (\rho-n) \]  
(6)

The quantity \( \varepsilon(c) = -\frac{u''(c)}{u'(c)c} \) is the elasticity of marginal utility with respect to consumption. It is constant for the utilities
\[ u(c) = \begin{cases} c^{1-\varepsilon} & 0 < \varepsilon < 1 \\ \ln c & \varepsilon = 1. \end{cases} \]

So for these utilities, (6) implies
\[ \frac{\dot{p}}{p} = -\varepsilon \frac{\dot{c}}{c} - (\rho-n). \]

On the other hand (3) implies
\[ \frac{\dot{p}}{p} = -(r-n) \]

Combining these yields
\[ \frac{\dot{c}}{c} = \frac{r - \rho}{\varepsilon} \]  
(7)

2 Firm’s problem

A firm wishes to maximize
\[ F(K, L) - rK - wL. \]

The first-order conditions are
\[ D_K F - r = 0, \quad D_L F - w = 0. \]

If \( F \) exhibits constant returns to scale we have
\[ r = f'(k), \quad w = f(k) - f'(k)k, \]
where \( k = K/L \) and \( f(k) = F(k,1) \).
3 Equilibrium

Substituting for $r$ in (7) yields

$$\frac{\dot{c}}{c} = \frac{f'(k) - \rho}{\varepsilon}$$  \hspace{1cm} (8)

and from (1) we have

$$\dot{k} = f(k) - nk - c.$$  \hspace{1cm} (9)

This pair of differential equations governs the evolution of the economy.

4 Phase diagram

This system of equations can be analyzed in terms of a *phase diagram*. Let $\hat{k}$ satisfy

$$f'(\hat{k}) = \rho.$$  

From (8), when $k = \hat{k}$, then $\dot{c} = 0$. The vertical line at $\hat{k}$ separates the plane into two regions. To the right, $\dot{c} < 0$, and to the left $\dot{c} > 0$. From (9), along the curve $c = f(k) - nk$, we have $\dot{k} = 0$. Above this curve $\dot{k} < 0$, and below
it, $\dot{k} > 0$. This curve intersects the vertical line at the point $(\hat{k}, \hat{c})$, which is a steady state. The arrows in Figure 1 indicate the direction of the time derivatives in these regions. The maximum sustainable level of $k$ is $\bar{k}$, where $f(\bar{k}) = n\bar{k}$.

The curves in Figure 2 show the trajectories of the mathematical solutions to the system (8)–(9). Not all of these solutions have an economic interpretation. The reason is that the system (8)–(9) is derived from a subset of the first order conditions for the producer and consumer. But it takes more than that to make an equilibrium. Above the green and blue curves, the paths $(k(t), c(t))$ have $k(t) \to 0$ and $c(t) \to \infty$, so these are clearly infeasible. Below the curves, $c(t) \to 0$ and $k(t) \to \bar{k}$. Thus $r(t) = f'(k(t)) \to f'(\bar{k}) < n < \rho$. Since $\dot{p}(t)/p(t) = -(r(t) - n)$, along these paths $p(t)$ grows, so the transversality condition $p(t)k(t) \to 0$ fails. Thus the equilibrium curves are the green and blue curves that converge monotonically to the steady state.

References

http://www.jstor.org/stable/2295827

http://www.jstor.org/stable/1879473
