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On the Cobb–Douglas Production Function

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In the 1920s the economist Paul Douglas was working on the problem of relating inputs and output at the national aggregate level. A survey by the National Bureau of Economic Research found that during the decade 1909–1918, the share of output payed to labor was fairly constant at about 74% (see the table in footnote 37 on page 163 of [1]), despite the fact the capital/labor ratio was not constant. He enquired of his friend Charles Cobb, a mathematician, if any particular production function might account for this. This gave birth to the original Cobb–Douglas production function

$$Y = A\,K^{1/4}L^{3/4},$$

which they propounded in their 1928 paper, "A Theory of Production" [1].

How did they know this was the answer?

Mathematically the problem is this: Assume that the formula $Y = F(K, L)$ governs relationship between output Y, capital K, and labor L. Assume that F is continuously differentiable. For every output price level p, wage rate w, and capital rental rate r, let $K^*(r, w, p)$ and $L^*(r, w, p)$ maximize profit,

$$pF(K, L) - rK - wL.$$

The first order conditions for an interior maximum are

$$pF_K(K^*, L^*) = r \tag{1}$$

$$pF_L(K^*, L^*) = w \tag{2}$$

where F_K denotes the partial derivative of F with respect to its first variable K, and F_L is with respect to L. Assume now that the fraction of output paid to labor is a constant α. For Cobb and Douglas they chose $\alpha = 0.75$. The constancy can be written:

$$(1 - \alpha)pF(K^*, L^*) = rK^* \tag{3}$$

$$\alpha p F(K^*, L^*) = wL^* \tag{4}$$

Dividing (1) by (3) gives

$$\frac{1}{K^*} = \frac{F_K(K^*, L^*)}{(1-\alpha)F(K^*, L^*)}. \tag{5}$$

We now use the chain rule to notice that $\frac{d}{dx}\ln\Big(f(x)\Big) = \frac{f'(x)}{f(x)}$ for any function f. This allows us to rewrite (5) as

$$\frac{\partial}{\partial K}\ln F = \frac{F_K}{F} = \frac{1-\alpha}{K^*}. \tag{6}$$

Similarly

$$\frac{\partial}{\partial L}\ln F = \frac{\alpha}{L^*}. \tag{7}$$

Thus we have eliminated p, r, and w. So the above equations hold for every (K^*, L^*) that can result as a profit maximum. If this is all of $\boldsymbol{R}^2_+$, then we may treat (6)–(7) as a system of partial differential equations that even I can solve. Since $\int \frac{1}{x} = \ln(x) + c$, where c is a constant of integration, we have

$$\ln F(K, L) = (1-\alpha)\ln K + g(L) + c, \tag{6$'$}$$

where $g(L)$ is a constant of integration that may depend on L; and

$$\ln F(K, L) = \alpha \ln L + h(K) + c', \tag{7$'$}$$

where $h(K)$ is a constant of integration that may depend on K. Combining these pins down $g(L)$ and $h(K)$, namely,

$$\ln F(K, L) = (1-\alpha)\ln K + \alpha \ln L + C$$

or, exponentiating both sides and letting $A = e^C$,

$$F(K, L) = AK^{1-\alpha}L^{\alpha}.$$

References

[1] Cobb, C. W. and P. H. Douglas. 1928. A theory of production. *American Economic Review* 18(1):139–165. Supplement, Papers and Proceedings of the Fortieth Annual Meeting of the American Economic Association.